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CHARACTERIZATION OF THE MATRIX WHOSE NORM IS DETERMINED BY ITS ACTION ON DECREASING SEQUENCES (THE EXCEPTIONAL CASES)

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Abstract

Let $A = (a_{j,k})_{j,k\geq 1}$ be a non-negative matrix. In this paper, we characterize those A for which $\|A\|_{\ell_p, \ell_q}$ are determined by their actions on non-negative decreasing sequences, where one of p and q is 1 or ∞. The conditions forcing on A are sufficient and they are also necessary for non-negative finite matrices.

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1. Introduction

For $x = \{x_k\}_{k=1}^{\infty}$, we write $x \geq 0$ if $x_k \geq 0$ for all k. We also write $x \downarrow$ for the case that $\{x_k\}_{k=1}^{\infty}$ is decreasing, that is, $x_k \geq x_{k+1}$ for all $k \geq 1$. For a matrix $A = (a_{j,k})_{j,k\geq 1}$, let $\|A\|_{E,F}$ denote the norm of A when $Ax = y$ defines an operator from $x \in E$ to $y \in F$, where $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ are two normed sequence spaces. More precisely, $\|A\|_{E,F} = \sup_{\|x\|_E = 1} \|Ax\|_F$. Clearly, $\|A\|_{E,F} \geq \|A\|_{E,F,\downarrow}$, where

$$\|A\|_{E,F,\downarrow} := \sup_{\|x\|_E = 1,\ x \geq 0,\, x\downarrow} \|Ax\|_F.$$

The study of $\|A\|_{E,F}$ has a long history in the literature and it goes back to the works of Hardy, Copson, and Hilbert (cf. [10]). In [10, Theorem 326], Hardy proved that $\|A\|_{\ell_p,\ell_p} = p/(p-1)$ for $1 < p < \infty$, where $A = (a_{j,k})_{j,k\geq 1}$ is the Cesàro matrix, defined by

$$a_{j,k} = \begin{cases} 1/j & \text{if} \quad k \leq j, \\ 0 & \text{if} \quad k > j. \end{cases}$$

This result can be restated in the following form, called the Hardy inequality:

$$\sum_{j=1}^{\infty} \left| \frac{1}{j} \sum_{k=1}^{j} x_k \right|^p \leq \left(\frac{p}{p-1} \right)^p \sum_{k=1}^{\infty} |x_k|^p \qquad (x = \{x_k\}_{k=1}^{\infty} \in \ell_p).$$

For general A, some of the related results can be found in [1], [3], [4], [6], [9], [14], and the references cited there. We also refer the readers to [5], [15], and [16] for the integral setting. As for the exceptional cases $p = 1$ or ∞, the readers can invoke [8], [11], [18], and others.

The question of determining $\|A\|_{E,F,\downarrow}$ was raised by Bennett (cf. [1, page 422] and [3, page 422]). In [3, Problem 7.23], Bennett asked the following upper bound problem for the case $E = F = \ell_p$: When does the equality $\|A\|_{E,F} = \|A\|_{E,F,\downarrow}$ hold? This problem has been partially solved by [1, page 422], [6, Lemma 2.4], and [12, Theorem 2]. Recently, in [7], the present authors gave a more general setting, which includes these as special cases. They characterized A and proved that E and F can be ℓ_p, $d(w, p)$, or $\ell_p(w)$, where $d(w, p)$ is the Lorentz sequence space associated with non-negative decreasing weights w_n and $\ell_p(w)$ consists of all sequences $x = \{x_k\}_{k=1}^{\infty}$ such that

$$\|x\|_{\ell_p}(w) := \left(\sum_{k=1}^{\infty} |x_k|^p w_k\right)^{1/p} < \infty.$$

However, the case $F = \ell_\infty$ is excluded in [7]. The main purpose of this paper is to deal with this case. In fact, we shall give a characterization of A for the case that $E = \ell_p$ and $F = \ell_q$, where one of p and q is equal to 1 or ∞. The details are given in Sections 2 and 3.

2. The Cases $p = 1$ or ∞

In this section, we investigate the upper bound equality $\|A\|_{\ell_p, \ell_q} = \|A\|_{\ell_p, \ell_q, \downarrow}$ for the cases $p = 1$ or ∞. The first main result is for $p = 1$.

Theorem 2.1. *Let* $1 \leq q \leq \infty$ *and* $A = (a_{j,k})_{j,k\geq 1}$ *with* $a_{j,k} \geq 0$. *Suppose that* $\|A\|_{\ell_1, \ell_q} < \infty$. *Then* (2.1) $\Leftrightarrow$ (2.2) $\Rightarrow$ (2.3), *where*

$$\left(\sum_{j=1}^{\infty} a_{j,1}^q\right)^{1/q} = \sup_{k\geq 1}\left(\sum_{j=1}^{\infty} a_{j,k}^q\right)^{1/q}, \tag{2.1}$$

$$\sup_{\|x\|_{\ell_1}=1} \|Ax\|_{\ell_q} = \max_{\|x\|_{\ell_1}=1,\, x\geq 0,\, x\downarrow} \|Ax\|_{\ell_q}, \tag{2.2}$$

$$\|A\|_{\ell_1, \ell_q} = \|A\|_{\ell_1, \ell_q, \downarrow}. \tag{2.3}$$

If in addition, $a_{j,k} = 0$ *for* $k > k_0$, *where* k_0 *is a given positive integer, then* (2.1)-(2.3) *are equivalent.*

Proof. By [8, Theorem 10] and [11, Equation (15)], we know that $\|A\|_{\ell_1, \ell_q} = \sup_{k\geq 1}\left(\sum_{j=1}^{\infty} a_{j,k}^q\right)^{1/q} < \infty$. Combining this with (2.1), we obtain

$$\sup_{\|x\|_{\ell_1}=1} \|Ax\|_{\ell_q} = \|A\|_{\ell_1, \ell_q} = \left(\sum_{j=1}^{\infty} a_{j,1}^q\right)^{1/q} = \|Ae_1\|_{\ell_q},$$

where $e_1 = (1, 0, \ldots)$ is decreasing. Hence, $(2.1) \Rightarrow (2.2)$. Assume that (2.2) holds. Then for some $x \geq 0$, we have $x \downarrow$, $\|x\|_{\ell_1} = 1$, and $\|Ax\|_{\ell_q} = \|A\|_{\ell_1, \ell_q}$. For such an x, it follows from [8, Theorem 10] and [11, Equation (15)] that

$$\|Ax\|_{\ell_q} = \|A\|_{\ell_1, \ell_q} = \sup_{k\geq 1}\left(\sum_{j=1}^{\infty} a_{j,k}^q\right)^{1/q} = \sup_{k\geq 1} S_k = M, \tag{2.4}$$

where $S_k = \left(\sum_{j=1}^{\infty} a_{j,k}^q\right)^{1/q}$ and $M = \sup_{k\geq 1} S_k$. For $1 \leq q < \infty$, the function $f(t) = t^q$ is convex on $[0, \infty)$. Hence, by the fact that $x_1 + \cdots + x_n + \cdots = 1$, we get

$$\begin{aligned} \|Ax\|_{\ell_q}^q &= (a_{1,1}x_1 + a_{1,2}x_2 + \cdots)^q + \cdots + (a_{n,1}x_1 + a_{n,2}x_2 + \cdots)^q + \cdots \\ &\leq x_1 S_1^q + x_2 S_2^q + \cdots + x_n S_n^q + \cdots \leq M^q. \end{aligned} \tag{2.5}$$

Putting (2.4)-(2.5) together yields $x_1 S_1^q + x_2 S_2^q + \cdots = M^q$, and consequently, $x_1(M^q - S_1^q) + x_2(M^q - S_2^q) + \cdots = 0$. We know that $x \geq 0$, $x \downarrow$, and $M^q - S_k^q \geq 0$ for all k. Therefore, $M = S_1$, that is, (2.1) holds. This establishes the equivalence $(2.1) \Leftrightarrow (2.2)$ for the case $1 \leq q < \infty$. For $q = \infty$, replace (2.5) by

$$\begin{aligned} \|Ax\|_{\ell_\infty} &= \sup_{j\geq 1}(a_{j,1}x_1 + a_{j,2}x_2 + \cdots) \\ &\leq x_1 S_1 + x_2 S_2 + \cdots + x_n S_n + \cdots \leq M, \end{aligned} \tag{2.6}$$

and modify the proof between (2.5) and (2.6). Then we shall get the equivalence (2.1) $\Leftrightarrow$ (2.2) for $q = \infty$. Clearly, (2.2) $\Rightarrow$ (2.3). It remains to prove the last conclusion. Assume that $a_{j,k} = 0$ for $j \geq 1$ and $k > k_0$. We shall prove

$$\sup_{\|x\|_{\ell_1} = 1, x \geq 0, x \downarrow} \|Ax\|_{\ell_q} \leq \|Ay\|_{\ell_q}, \tag{2.7}$$

for some y with $y \geq 0$, $y \downarrow$, and $\|y\|_{\ell_1} = 1$. If so, then (2.3) implies (2.2) and we are done. We have $\|A\tilde{x}\|_{\ell_q} \geq \|Ax\|_{\ell_q}$, where $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \cdots)$, $\tilde{x}_1 = x_1 + (x_{k_0+1} + x_{k_0+2} + \cdots)$, $\tilde{x}_k = x_k$ for $1 < k \leq k_0$, and $\tilde{x}_k = 0$ otherwise. Hence, this substitution does not loose the value of the left-hand side of (2.7). Without loss of generality, the sequences x and y in (2.7) will be assumed to be of the form $\tilde{\xi} = (\xi_1, \cdots, \xi_{k_0}, 0, \cdots)$. Set $\xi^* = (\xi_1, \cdots, \xi_{k_0})$ and $\|\xi^*\|_{\ell_1} = \sum_{k=1}^{k_0} |\xi_k|$. We know that the set $\Omega = \{\xi^* : \tilde{\xi} \geq 0, \tilde{\xi} \downarrow, \text{ and } \|\tilde{\xi}\|_{\ell_1} = 1\}$ is a non-empty compact subset of $\mathbb{R}^{k_0}$ and the mapping $\tilde{A} : \Omega \mapsto \mathbb{R}$ is continuous, where $\tilde{A}\xi^* = \|A\tilde{\xi}\|_{\ell_q}$. Hence, the sequence y involved in (2.7) exists. This completes the proof of Theorem 2.1.

We know that $\|A\|_{\ell_1, \ell_q} = \sup_{k \geq 1} \left(\sum_{j=1}^{\infty} a_{j,k}^q \right)^{1/q}$, so the condition $\|A\|_{\ell_1, \ell_q} < \infty$ in Theorem 2.1 can be replaced by the statement that the quantity on the right side of (2.1) is finite. For a finite matrix, $a_{j,k} = 0$ for $\max(j, k) > k_0$, where k_0 exists. Moreover, $\|A\|_{l_1, l_q} < \infty$. Hence, (2.1)-(2.3) in Theorem 2.1 are equivalent for this case. In general, (2.3) does not imply (2.1). A counterexample is given by the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1/4 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1/4 & 1 & 0 & \cdots \\ 0 & 0 & 1/9 & 1/4 & 1 & \cdots \\ 0 & 0 & 0 & 1/9 & 1/4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

For $x \geq 0$ with $\|x\|_{\ell_1} = 1$, we have

$$\|Ax\|_{\ell_1} = \sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{j,k}\right)x_k \leq \frac{\pi^2}{6}\left(\sum_{k=1}^{\infty} x_k\right) = \frac{\pi^2}{6}.$$

This implies $\|A\|_{\ell_1, \ell_1} \leq \pi^2/6$. On the other hand, the choice $x_n = (\frac{1}{n}, \ldots, \frac{1}{n}, 0, \ldots)$ gives $x_n \geq 0$, $x_n \downarrow$, $\|x_n\|_{\ell_1} = 1$, and

$$\begin{aligned} \|Ax_n\|_{\ell_1} &= \frac{1}{n}\left(\sum_{j=1}^{\infty} a_{j,1} + \cdots + \sum_{j=1}^{\infty} a_{j,n}\right) \\ &= \frac{1}{n}\left(1 + (1 + \frac{1}{4}) + \cdots + (1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2})\right) \\ &\to 1 + \frac{1}{4} + \frac{1}{9} + \cdots = \frac{\pi^2}{6} \qquad \text{as } n \to \infty. \end{aligned}$$

This leads us to $\|A\|_{\ell_1, \ell_1} = \pi^2/6 = \sup_{\|x\|_{\ell_1}=1,\, x \geq 0,\, x\downarrow} \|Ax\|_{\ell_1}$, which says that (2.3) holds for $q = 1$. However, we can easily see that (2.1) is false for $q = 1$.

The next theorem deals with the case $p = \infty$.

Theorem 2.2. *Let* $1 \leq q \leq \infty$ *and* $A = (a_{j,k})_{j,k\geq 1}$ *with* $a_{j,k} \geq 0$. *Then*

$$\|A\|_{\ell_\infty, \ell_q} = \left(\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty} a_{j,k}\right)^q\right)^{1/q} = \|A\|_{\ell_\infty, \ell_q, \downarrow}. \tag{2.8}$$

Proof. Consider $1 \le q < \infty$. For $x \ge 0$ with $\|x\|_{\ell_\infty} = 1$, we have

$$\|Ax\|_{\ell_q} = \left(\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty} a_{j,k}x_k\right)^q\right)^{1/q} \le \left(\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty} a_{j,k}\right)^q\right)^{1/q},$$

and the right-hand side of the above inequality is attained by $x = (1, 1, \ldots)$. Therefore, (2.8) holds for $1 \le q < \infty$. As for $q = \infty$,

$$\|Ax\|_{\ell_\infty} = \sup_{j\ge1}\left(\sum_{k=1}^{\infty} a_{j,k}x_k\right) \le \sup_{j\ge1}\left(\sum_{k=1}^{\infty} a_{j,k}\right),$$

where $x \ge 0$ and $\|x\|_{\ell_\infty} = 1$. Moreover, the choice $x = (1, 1, \ldots)$ gives $\|Ax\|_{\ell_\infty} = \sup_{j\ge1}\left(\sum_{k=1}^{\infty} a_{j,k}\right)$. Hence, (2.8) holds for $q = \infty$ and the proof is complete.

From (2.8) and the proof of Theorem 2.2, we see that $\|A\|_{\ell_\infty,\ell_q} < \infty$ if and only if $\left(\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty} a_{j,k}\right)^q\right)^{1/q} < \infty$. Moreover, under this condition, the following equality also holds:

$$\sup_{\|x\|_{\ell_\infty}=1}\|Ax\|_{\ell_q} = \max_{\|x\|_{\ell_\infty}=1,\ x\ge0,\ x\downarrow}\|Ax\|_{\ell_q}. \tag{2.9}$$

3. The Cases $q = 1$ or ∞

In this section, we investigate the upper bound equality for the cases $q = 1$ or ∞. Since $p = 1$ or ∞ have been examined in Theorems 2.1-2.2, we exclude these two cases in the following, that is, we only consider the case $1 < p < \infty$.

Theorem 3.1. *Let* $1 < p < \infty$ *and* $A = (a_{j,k})_{j,k\ge1}$ *with* $a_{j,k} \ge 0$. *Suppose that* $\|A\|_{\ell_p,\ell_1} < \infty$. *Then* (3.1) $\Leftrightarrow$ (3.2) $\Leftrightarrow$ (3.3), *where*

$$\sum_{j=1}^{\infty} a_{j,k} \text{ is decreasing in } k, \tag{3.1}$$

$$\sup_{\|x\|_{\ell_p}=1} \|Ax\|_{\ell_1} = \max_{\|x\|_{\ell_p}=1,\, x\geq 0,\, x\downarrow} \|Ax\|_{\ell_1}, \tag{3.2}$$

$$\|A\|_{\ell_p,\ell_1} = \|A\|_{\ell_p,\ell_1,\downarrow}. \tag{3.3}$$

Proof. By [8, page 699, Corollary 1], we know that

$$\|A\|_{\ell_p,\ell_1} = \left(\sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{j,k}\right)^{p^*}\right)^{1/p^*} < \infty, \tag{3.4}$$

where $1/p + 1/p^* = 1$. Set $S_k = \sum_{j=1}^{\infty} a_{j,k}$. Then (3.1) says that $\{S_k\}_{k=1}^{\infty}$ is decreasing. Let $x = (x_1, x_2, \dots)$, where $x_k = \lambda S_k^{p^*-1}$ and $\lambda = \left(\sum_{k=1}^{\infty} S_k^{p^*}\right)^{-1/p}$. Then $x \geq 0$, $x\downarrow$, $\|x\|_{\ell_p} = 1$, and

$$\|Ax\|_{\ell_1} = \left(\sum_{k=1}^{\infty} S_k^{p^*}\right)^{1/p^*} = \left(\sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{j,k}\right)^{p^*}\right)^{1/p^*} = \|A\|_{\ell_p,\ell_1}.$$

Hence, (3.1) $\Rightarrow$ (3.2). Clearly, (3.2) $\Rightarrow$ (3.3). We claim that (3.3) $\Rightarrow$ (3.2) $\Rightarrow$ (3.1). Assume that (3.3) holds. By (3.4), $\|A\|_{\ell_p,\ell_1} = \left(\sum_{k=1}^{\infty} S_k^{p^*}\right)^{1/p^*}$, and so there exists some $x^n = (x_1^n, x_2^n, x_3^n, \dots) \in \ell_p$ such that $x^n \geq 0$, $x^n\downarrow$, $\|x^n\|_{\ell_p} = 1$, and $\|Ax^n\|_{\ell_1} \to \left(\sum_{k=1}^{\infty} S_k^{p^*}\right)^{1/p^*}$ as $n \to \infty$. We know that $\{x_k^n : n \geq 1\} \subset [0, 1]$ for each k. By the "diagonal process" (cf. [17, Theorem 7.23]), without loss of generality, we can further assume that for each $k \geq 1$, x_k^n converges to some $\tilde{x}_k$ as $n \to \infty$. Set $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \cdots)$. Then $\tilde{x} \geq 0$ and $\tilde{x}\downarrow$. We shall claim that $\|\tilde{x}\|_{\ell_p} = 1$ and $\|A\tilde{x}\|_{\ell_1} = \|A\|_{\ell_p,\ell_1}$. If so, (3.2) follows. For any $m \geq 1$, we have

$$\left(\sum_{k=1}^{m} \tilde{x}_k^p\right)^{1/p} = \lim_{n\to\infty}\left(\sum_{k=1}^{m} \left(x_k^n\right)^p\right)^{1/p} \leq \lim_{n\to\infty}\|x^n\|_{\ell_p} = 1,$$

which implies $\|\tilde{x}\|_{\ell_p} \leq 1$. We shall prove $\|\tilde{x}\|_{\ell_p} \geq 1$ and $\|A\tilde{x}\|_{\ell_1} = \|A\|_{l_p, l_1}$ simultaneously. By definitions, $\|A\tilde{x}\|_{\ell_1} = \sum_{k=1}^{\infty} S_k \tilde{x}_k$ and $\|Ax^n\|_{\ell_1} = \sum_{k=1}^{\infty} S_k x_k^n$. For $m \geq 1$, it follows from the Hölder inequality that

$$\left|\sum_{k=1}^{m} S_k \tilde{x}_k - \sum_{k=1}^{\infty} S_k x_k^n\right| \leq \left|\sum_{k=1}^{m} S_k (\tilde{x}_k - x_k^n)\right| + \|x^n\|_{\ell_p}\left(\sum_{k=m+1}^{\infty} S_k^{p^*}\right)^{1/p^*}$$

$$= \left|\sum_{k=1}^{m} S_k (\tilde{x}_k - x_k^n)\right| + \left(\sum_{k=m+1}^{\infty} S_k^{p^*}\right)^{1/p^*}.$$

This implies

$$\sum_{k=1}^{m} S_k \tilde{x}_k \geq \|Ax^n\|_{\ell_1} - \left|\sum_{k=1}^{m} S_k (\tilde{x}_k - x_k^n)\right| - \left(\sum_{k=m+1}^{\infty} S_k^{p^*}\right)^{1/p^*}$$

$$\to \left(\sum_{k=1}^{\infty} S_k^{p^*}\right)^{1/p^*} - \left(\sum_{k=m+1}^{\infty} S_k^{p^*}\right)^{1/p^*} \quad \text{as} \quad n \to \infty.$$

Taking $m \to \infty$, we get $\|A\tilde{x}\|_{\ell_1} \geq \left(\sum_{k=1}^{\infty} S_k^{p^*}\right)^{1/p^*}$. For the reverse inequality, by the Hölder inequality and $\|\tilde{x}\|_{\ell_p} \leq 1$, we obtain

$$\|A\tilde{x}\|_{\ell_1} = \sum_{k=1}^{\infty} S_k \tilde{x}_k \leq \|\tilde{x}\|_{\ell_p}\left(\sum_{k=1}^{\infty} S_k^{p^*}\right)^{1/p^*} \leq \left(\sum_{k=1}^{\infty} S_k^{p^*}\right)^{1/p^*}. \tag{3.5}$$

Putting these inequalities together, we find that $\|A\tilde{x}\|_{\ell_1} = \left(\sum_{k=1}^{\infty} S_k^{p^*}\right)^{1/p^*} = \|A\|_{\ell_p, \ell_1}$ and $\|\tilde{x}\|_{\ell_p} = 1$. This finishes the proof of the implication: (3.3) $\Rightarrow$ (3.2). In fact, we get more. Since the inequality signs in (3.5) are equality signs. By the Hölder inequality, we infer that

$(\tilde{x}_1^p, \tilde{x}_2^p, \dots)$ and $(S_1^{p^*}, S_2^{p^*}, \dots)$ are proportional. Since $\tilde{x}_1^p \geq \tilde{x}_2^p \geq \cdots$, the sequence $\{S_k^{p^*}\}_{k=1}^{\infty}$ is decreasing. This leads us to (3.1). We complete the proof.

We know that $\|A\|_{\ell_p, \ell_1} = \left(\sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{j,k}\right)^{p^*}\right)^{1/p^*}$. Hence, the condition $\|A\|_{\ell_p, \ell_1} < \infty$ in Theorem 3.1 can be replaced by $\left(\sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{j,k}\right)^{p^*}\right)^{1/p^*} < \infty$. As Theorems 2.1-2.2 indicate, Theorem 3.1 is false for the cases that $p = 1$ or ∞.

In [7], the present authors indicate that the matrix A, defined by $a_{2,2} = 1$ and 0 otherwise, possesses the property: $\|A\|_{\ell_p, \ell_\infty} > \|A\|_{\ell_p, \ell_\infty, \downarrow}$, where $1 \leq p < \infty$. This phenomenon can be interpreted by applying the following result to the case $\Lambda = \{2\}$.

Theorem 3.2. *Let* $1 < p < \infty, 1/p + 1/p^* = 1$, *and* $A = (a_{j,k})_{j,k \geq 1}$ *with* $a_{j,k} \geq 0$. *Suppose that there exists a nonempty finite set* Λ *of positive integers with*

$$\sup_{j \notin \Lambda}\left(\sum_{k=1}^{\infty} a_{j,k}^{p^*}\right)^{1/p^*} < \sup_{j \in \Lambda}\left(\sum_{k=1}^{\infty} a_{j,k}^{p^*}\right)^{1/p^*} < \infty. \tag{3.6}$$

Then (3.7) $\Leftrightarrow$ (3.8) $\Leftrightarrow$ (3.9), *where*

there exists some $l \in \Lambda$ *such that* $a_{l,1} \geq a_{l,2} \geq \cdots \geq a_{l,n} \geq \cdots$ *and*

$$\left(\sum_{k=1}^{\infty} a_{l,k}^{p^*}\right)^{1/p^*} = \sup_{j \geq 1}\left(\sum_{k=1}^{\infty} a_{j,k}^{p^*}\right)^{1/p^*}, \tag{3.7}$$

$$\sup_{\|x\|_{\ell_p}=1} \|Ax\|_{\ell_\infty} = \max_{\|x\|_{\ell_p}=1, x \geq 0, x \downarrow} \|Ax\|_{\ell_\infty}, \tag{3.8}$$

$$\|A\|_{\ell_p, \ell_\infty} = \|A\|_{\ell_p, \ell_\infty, \downarrow}. \tag{3.9}$$

For the implication from (3.7) *to any of* (3.8) *or* (3.9), *the condition that* Λ *is finite is unnecessary.*

Proof. Putting the Hellinger-Toeplitz theorem (see [2, page 29]), [8, Theorem 10], and (3.6) together, we obtain

$$\|A\|_{\ell_p, \ell_\infty} = \|A^t\|_{\ell_1, \ell_{p^*}} = \sup_{j \geq 1} \left(\sum_{k=1}^{\infty} a_{j,k}^{p^*} \right)^{1/p^*} < \infty, \tag{3.10}$$

where A^t is the transpose of A. Assume that (3.7) holds. Set $x = (x_1, x_2, \dots)$, where $x_k = \lambda a_{l,k}^{p^*-1}$ and $\lambda = \left(\sum_{k=1}^{\infty} a_{l,k}^{p^*} \right)^{-1/p}$. Then $x \geq 0$, $x \downarrow$, $\|x\|_{\ell_p} = 1$, and

$$\|Ax\|_{\ell_\infty} \geq \sum_{k=1}^{\infty} a_{l,k} x_k = \left(\sum_{k=1}^{\infty} a_{l,k}^{p^*} \right)^{1/p^*}. \tag{3.11}$$

By (3.7) and (3.10), we get $\|Ax\|_{\ell_\infty} \geq \|A\|_{\ell_p, \ell_\infty}$. This leads us to (3.8). Clearly, (3.8) $\Rightarrow$ (3.9). In the above argument, the assumption that Λ is finite is unnecessary. We claim that (3.9) $\Rightarrow$ (3.7). Assume that (3.9) holds. We know that Λ is a finite set. Without loss of generality, we can assume that $\left(\sum_{k=1}^{\infty} a_{r,k}^{p^*} \right)^{1/p^*} = \sup_{j \in \Lambda} \left(\sum_{k=1}^{\infty} a_{j,k}^{p^*} \right)^{1/p^*}$ for all $r \in \Lambda$. Let $x \geq 0$, $x \downarrow$, $\|x\|_{\ell_p} = 1$, and $\|Ax\|_{\ell_\infty} > \gamma$, where $\gamma = \sup_{j \notin \Lambda} \left(\sum_{k=1}^{\infty} a_{j,k}^{p^*} \right)^{1/p^*}$. We have $\|Ax\|_{\ell_\infty} = \sup_{j \geq 1} \left(\sum_{k=1}^{\infty} a_{j,k} x_k \right)$. For $j \geq 1$, the Hölder inequality implies

$$\sum_{k=1}^{\infty} a_{j,k} x_k \leq \|x\|_{\ell_p} \left(\sum_{k=1}^{\infty} a_{j,k}^{p^*} \right)^{1/p^*} = \left(\sum_{k=1}^{\infty} a_{j,k}^{p^*} \right)^{1/p^*},$$

which gives $\sup_{j \notin \Lambda}\left(\sum_{k=1}^{\infty} a_{j,k}x_k\right) \leq \sup_{j \notin \Lambda}\left(\sum_{k=1}^{\infty} a_{j,k}^{p^*}\right)^{1/p^*} = \gamma$. Thus, $\|Ax\|_{\ell_\infty} = \sum_{k=1}^{\infty} a_{r,k}x_k$ for some $r \in \Lambda$. Since Λ is a finite set, we can find some $l \in \Lambda$ such that

$$\sup_{\|x\|_{\ell_p}=1,\, x\geq 0,\, x\downarrow} \sum_{k=1}^{\infty} a_{l,k}x_k = \|A\|_{\ell_p,\, \ell_\infty,\downarrow}. \tag{3.12}$$

Putting (3.6), (3.9), (3.10), and (3.12) together yields

$$\sup_{\|x\|_{\ell_p}=1,\, x\geq 0,\, x\downarrow} \sum_{k=1}^{\infty} a_{l,k}x_k = \left(\sum_{k=1}^{\infty} a_{l,k}^{p^*}\right)^{1/p^*},$$

which can be written in the form: $\|\widetilde{A}\|_{\ell_p,\ell_1} = \|\widetilde{A}\|_{\ell_p,\ell_1,\downarrow}$. Here $\widetilde{A} = (\widetilde{a}_{j,k})_{j,k\geq 1}$ is defined by $\widetilde{a}_{l,k} = a_{l,k}$ and $\widetilde{a}_{j,k} = 0$ for $j \neq l$. By Theorem 3.1, we get (3.7). The proof is complete.

From (3.10), we see that condition (3.6) implies $\|A\|_{\ell_p,\ell_\infty} < \infty$. It is clear that this condition is automatically satisfied by any finite non-negative matrix A. Applying Theorem 3.2 to this case, we find that (3.7)-(3.9) are equivalent for such kind of matrices. In general, (3.6) can not be taken off. The following matrix provides us a counterexample:

$$A = (a_{j,k})_{j,k\geq 1} = \begin{pmatrix} 1/2 & 1 & 1/3 & 1/4 & 1/5 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1/2 & 0 & 0 & 0 & \cdots \\ 1 & 1/2 & 1/3 & 0 & 0 & \cdots \\ 1 & 1/2 & 1/3 & 1/4 & 0 & \cdots \\ 1 & 1/2 & 1/3 & 1/4 & 1/5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Clearly, both of (3.6)-(3.7) are not satisfied by any finite set Λ. Let $x_k = (1/k)^{p^*-1}\left(\sum_{s=1}^{n}(1/s)^{p^*}\right)^{-1/p}$ for $1 \le k \le n$ and 0 otherwise, where $1 < p < \infty$. Then $x \ge 0$, $x \downarrow$, $\|x\|_{\ell_p} = 1$, and $\|Ax\|_{\ell_\infty} \ge \sum_{k=1}^{n} x_k / k = \left(\sum_{k=1}^{n}(1/k)^{p^*}\right)^{1/p^*}$ for $n \ge 2$. This leads us to

$$\|A\|_{\ell_p,\ell_\infty,\downarrow} \ge \|Ax\|_{\ell_\infty} \ge \left(\sum_{k=1}^{n}(1/k)^{p^*}\right)^{1/p^*} = \left(\sum_{k=1}^{\infty} a_{n+1,k}^{p^*}\right)^{1/p^*} \quad (n \ge 2).$$

Putting this with (3.10) and letting $n \to \infty$, we obtain $\|A\|_{\ell_p,\ell_\infty,\downarrow} \ge \|A\|_{\ell_p,\ell_\infty}$. Hence, (3.9) holds.

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